Characteristic Impedances of Generalized Rectangular Transmission Lines

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Abstract—**The impedance of the generalized strip transmission line is computed by variational methods. The use of an upper bound and lower bound approximation yields an average impedance as well as a known maximum error. The one-dielectric microstrip line is treated as a limiting case. Losses are considered. Results include those problems which have been solved elsewhere by conformal mapping techniques as special cases.**

Introduction

THE CHARACTERISTIC impedance of the strip transmission line has been discussed by many authors. The design of transmission line components, investigation of mechanical tolerances and other applications require the solution of the generalized strip transmission line, i.e., the impedance of a line formed by a conductor placed parallel to, but at arbitrary distances from the two ground planes must be available. The microstrip transmission line would be included as a limiting case in this type of configuration. An extension of a known variational technique [1] is chosen as a tool for the desired mathematical model.

Transverse Plane Capacitance

An upper bound on the capacitance may be obtained by considering the stationary property of the energy integral. Thus,

$$C = \frac{1}{V_0^2}\,\epsilon \int\int_{\Sigma} \bar{\nabla}_t\phi \cdot \bar{\nabla}_t\phi\, dx dy \tag{1}$$

The integral is stationary with respect to first-order variations in ϕ, where ϕ is the solution of Laplace's equation for the boundary conditions and geometry given in Fig. 2. The total energy stored in the regions is given by:

$$W_T = \frac{1}{4}\epsilon_1\left[\left(\frac{W}{A}+\frac{W}{B}\right)V_0^2 + \frac{\pi}{2}\sum_{n=1}^{\infty} n\left(a_n^2 \sinh\frac{n\pi W}{A} + b_n^2 \sinh\frac{n\pi W}{B} + 2c_n^2 e^{-n\pi W}\right)\right]. \tag{2}$$

The coefficients may be evaluated in terms of the potential $\phi(W/2, y)$. It is possible to express this quantity as

$$\phi\left(\frac{W}{2}, y\right) = \left(V_0\frac{y}{A} + g(y)\right)((U(y) - U(y-A)) + V_0(U(y-A) - U(y-A-T)) + \left(V_0\frac{1-y}{B} + f(y)\right)(U(y-A-T) - U(y-1)) \tag{3}$$

where $U(\alpha)$ is the step function defined by

$$U(\alpha) = {1 \atop 0} \quad \text{if } \alpha \gtrless 0.$$

The assumption that $g(y)$ and $f(y)$ are zero is used for the first-order approximation. This implies that only c_n has a nonzero value. Since the stationary value of the integral is a minimum, it follows that the value of the capacitance calculated with these approximations will always be larger than the exact value. This upper bound is found to be:

$$C^U = \epsilon_1\left[\frac{W}{A} + \frac{W}{1-A-T} + 4\sum_{n=1}^{\infty}\frac{1}{n\pi}(S(n\pi A) + (-1)^{n+1}S(n\pi B))\right] \tag{4}$$

where

$$S(\alpha) = \frac{1}{\alpha}\sin\alpha.$$

It is possible to calculate a lower bound on the capacitance by making use of the Green's function approach. The center conductor is removed and replaced by an equivalent surface charge ρ. The potential is then given by

$$\phi(x, y) = \oint_{S_2} G(x, y \mid x', y')\rho(x', y')dl' \tag{5}$$

where S_2 is the surface of the center conductor, and G is the solution of

$$\bar{\nabla}_t^2 G(x, y \mid x', y') = -\frac{1}{\epsilon_1}\delta(x - x')\delta(y - y')$$

$$G(x, 0 \mid x', y') = 0$$

$$G(x, 1 \mid x', y') = 0.$$

The capacitance is:

$$\frac{1}{C} = \frac{\oint_{S_2}\oint_{S_2} G(x, y \mid x', y')\rho(x'y')\rho(x, y)dldl'}{\left[\oint_{S_2}\rho(x, y)dl\right]^2}. \tag{6}$$

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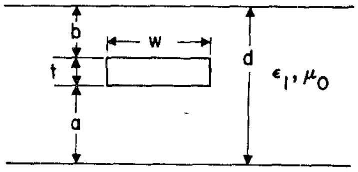

Fig. 1. Generalized rectangular transmission line.

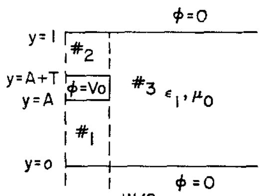

Fig. 2. Boundary values and definition of coordinates.

A relationship between α and α' may be obtained by requiring that the surface charge density is a constant when $A=B$. Thus:

$$\rho = \frac{q_{\text{side}}}{2T} = \alpha = \frac{q_{\text{length}}}{2W} = \frac{\alpha'}{A}$$

Hence

$$\left[\oint \rho dl\right]^2 = 4\alpha^2\left[T + \frac{W}{2}\left(1 + \frac{A}{B}\right)\right]^2. \tag{8}$$

Since α will appear in the numerator and denominator of an expression similar to that given by (6), it will cancel from the final expression. The rather lengthy integration of (6) is straight forward and will not be given here. The final result is given by:

$$\begin{aligned}\frac{1}{C^L} = \sum_{n=1}^{\infty}&\left[\frac{1}{\epsilon}\left(\frac{A}{T+\frac{W}{2}\left(1+\frac{A}{1-A-T}\right)}\right)\right]^2\left[S(n\pi A) + (-1)^{n+1}S(n\pi(1-A-T))\right]\\ &\cdot\left[(S(n\pi A) + (-1)^{n+1}S(n\pi(1-A-T)))\left(\frac{W}{2} - \frac{1}{2\pi n}(1-\epsilon^{-n\pi W})\right)\right.\\ &\left.+\frac{1}{n^2\pi^2 A}\sin\left(n\pi\left(A+\frac{T}{2}\right)\right)\sin\left(\frac{n\pi T}{2}\right)(1-\epsilon^{-n\pi W})\right] + \frac{1}{\epsilon}\,\frac{A}{\left(T+\frac{W}{2}\left(1+\frac{A}{1-A-T}\right)\right)^2}\,\frac{1}{n^2\pi^2}\\ &\cdot\left[\sin\left(n\pi\left(A+\frac{T}{2}\right)\right)\sin\left(\frac{n\pi T}{2}\right)\right]\left[(S(n\pi A) + (-1)^{n+1}S(n\pi(1-A-T)))(1-\epsilon^{-n\pi W})\right.\\ &\left.+\frac{2}{n\pi A}\sin\left(n\pi\left(A+\frac{T}{2}\right)\right)\sin\left(\frac{n\pi T}{2}\right)(1+\epsilon^{-n\pi W})\right].\end{aligned} \tag{9}$$

It may be shown that this integral is stationary with respect to first-order changes in the functional $\rho(x, y)$. The function G is the Green's function for the situation described by Fig. 1. It may be shown to be:

$$G = \frac{1}{n\epsilon}\sum_n \frac{1}{n}\sin\frac{n\pi y'}{d}\sin\frac{n\pi y}{d}\,\epsilon^{-(n\pi/d)|x-x'|}. \tag{7}$$

The surface density is of course unknown. If it is assumed that ρ_s has some reasonable form, a value of C, say C^L, may be computed. Since the stationary value of the integral is a minimum, it follows that $C^L < C_{\text{exact}}$. Thus, the assumption that ρ is a constant on all conductor surfaces and inversely proportional to the ground plane spacing leads to the following:

$$\oint_{S_2} \rho dl = 2\alpha T + \alpha'\left(\frac{W}{B} + \frac{W}{A}\right).$$

TEM Mode Transmission Lines

For a homogeneous cross section the characteristic impedance of the corresponding transmission line may be expressed as

$$Z_0 = \sqrt{\epsilon_R}\,\frac{\sqrt{\mu_0\epsilon_0}}{C}$$

and since

$$C^U \geqq C_{\text{exact}} \geqq C^L$$

$$Z_0{}^L = \frac{\sqrt{\mu_0\epsilon}}{C^U} \leqq (Z_0)_{\text{exact}} \leqq \frac{\sqrt{\mu_0\epsilon}}{C^L} \equiv Z_0{}^U \tag{10}$$

simplification of (4) and (9) yields:

$$Z_0^L = \frac{\eta/\sqrt{\epsilon_R}}{\dfrac{W}{A} + \dfrac{W}{1-A-T} + 4\sum_{n=1}^{\infty} \dfrac{1}{n\pi}\left[S(n\pi A) + (-1)^{n+1}S(n\pi(1-A-T))\right]^2} \tag{11}$$

$$Z_0^U = \frac{\eta}{\epsilon_R}\left[\frac{A}{T + \dfrac{W}{2}\left(1 + \dfrac{A}{1-A-T}\right)}\right]^2 \Bigg[\left(S(n\pi A) + (-1)^{n+1}S(n\pi(1-A-T))\right)^2\left(\frac{W}{2} - \frac{1}{2\pi n}\left(1 - \epsilon^{-n\pi W}\right)\right)$$
$$+ \frac{2}{n\pi A}\left(\sin\left(n\pi\left(A + \frac{T}{2}\right)\right)\sin\left(\frac{n\pi T}{2}\right)\right)^2\left(1 + \epsilon^{-n\pi W}\right)$$
$$+ \frac{2}{n^2\pi^2 A}\left(\sin\left(n\pi\left(A + \frac{T}{2}\right)\right)\sin\left(\frac{n\pi T}{2}\right)\right)\left(S(n\pi A) + (-1)^{n+1}S(n\pi(1-A-T))\right)\left(1 - \epsilon^{-n\pi W}\right)\Bigg] \tag{12}$$

where

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0}}.$$

The complexity of Z_0^U is due to the charge distribution assumption. A somewhat simpler expression may be obtained by assuming the distribution to be a constant. This approximation is discussed in a later section. An average impedance is defined by (13).

$$\overline{Z}_0 = \tfrac{1}{2}(Z_0^L + Z_0^U). \tag{13}$$

Figures 3 and 4 represent the impedance surfaces for two different conductor thicknesses and a range of ground plane spacings as well as conductor widths. Since the line geometry is quite flexible, it is possible to obtain the TEM impedance of the microstrip transmission line as a limiting case. Furthermore, the loading effect of the top ground plane may be studied. Thus, if

$$Z_0(a_i, b_i, t_i, w_i) \rightarrow Z_0(a_i, nb_i, t_i, w_i)$$

the corresponding transform in the normalized coordinate system is:

$$Z_0(A_i, T_i, W_i) \rightarrow Z_0(kA_i, kT_i, kW_i)$$

where

$$k = \frac{1}{n + (1-n)(A_i + T_i)}.$$

This transformation was used to obtain the data given in Fig. 5. It is seen that the line impedance for this geometry goes from 21.3 ohms to 33.5 ohms for the transition from symmetrical strip line to microstrip. The change is roughly 50 per cent. Furthermore, if the ground planes are separated in a ratio of 10 to 1 or greater, the loading effect of the upper ground plane disappears and the line may be considered as a microstrip line. The increase in impedance is, of course, a function of the center conductor width. Increases of 85 per cent were observed for very wide lines ($W=60$). Smaller lines ($W=1.2$) showed increases of the order of 25 per cent.

The data displayed in Fig. 6 was obtained by the forementioned process. It agrees very well with evaluations made for $T=0$ by Bowman [3] and Black and Higgins [2].

Loss Evaluation

It has been shown that the attenuation constant for the series losses in a transmission line (α_c) is approximately given by [4], [5]:

$$\alpha_c = \frac{R_s\sqrt{\epsilon_r}}{2\eta}\,\frac{1}{Z_0}\left(\frac{\delta Z_0}{\delta n}\right) \text{ nepers/unit length} \tag{14}$$

where

R_s = surface resistivity (ohm/square)

δZ_0 = incremental increase in characteristic impedance due to a uniform incremental decrease in dimension δn of all conductors normal to their surface.

The equation is a reasonable approximation for frequencies which are high enough to cause fully developed skin effect for a given conductor thickness. It is shown that in the unnormalized coordinate system, (15) applies

$$\frac{\delta Z_0}{\delta n} = 2\left(\frac{\partial Z_0}{\partial d} - \frac{\partial Z_0}{\partial w} - \frac{\partial Z_0}{\partial t}\right). \tag{15}$$

The expression may be transformed to yield a similar answer in the normalized system used here. In order to do this some care must be taken with the term $\partial Z_0/\partial d$. A limit process was found to be helpful. It was found that:

$$\left(\frac{\delta Z_0}{\delta n}\right)_{CT} = \frac{1}{d}\left[\frac{\partial Z_0}{\partial A} - 2\frac{\partial Z_0}{\partial T} - 2\frac{\partial Z_0}{\partial W}\right] \tag{16a}$$

$$\left(\frac{\delta Z_0}{\delta n}\right)_a = \frac{1}{d}\left[(1-A)\frac{\partial Z_0}{\partial A} - T\frac{\partial Z_0}{\partial T} - W\frac{\partial Z_0}{\partial W}\right] \tag{16b}$$

$$\left(\frac{\delta Z_0}{\delta n}\right)_b = -\frac{1}{d}\left[A\frac{\partial Z_0}{\partial A} + T\frac{\partial Z_0}{\partial T} + W\frac{\partial Z_0}{\partial W}\right] \tag{16c}$$

where the three expressions pertain to the center conductor, the lower ground plane, and the upper ground plane, respectively. If all conductors involve the same material, it is found that:

$$\frac{2\eta\alpha_c d}{R_s\sqrt{\epsilon_R}} = \frac{1}{Z_0}\left[(1-A)\frac{\partial Z_0}{\partial A} - (1+T)\frac{\partial Z_0}{\partial T} - (1+W)\frac{\partial Z_0}{\partial W}\right]. \quad (17)$$

Equation (17) has been evaluated for a variety of cases. Some of these are given in Fig. 7.

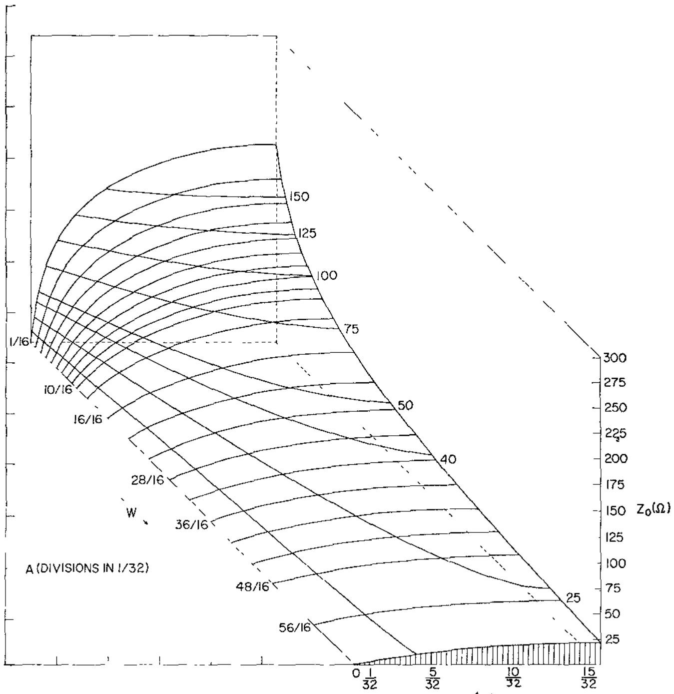

Fig. 3. Impedance surface for $T=0$ and $\epsilon_R=1$.

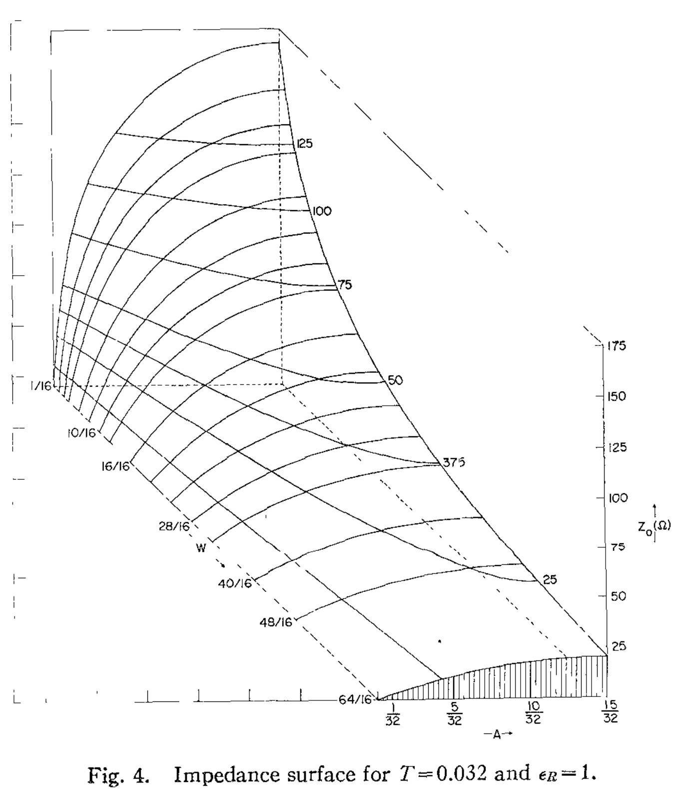

Fig. 4. Impedance surface for $T=0.032$ and $\epsilon_R=1$.

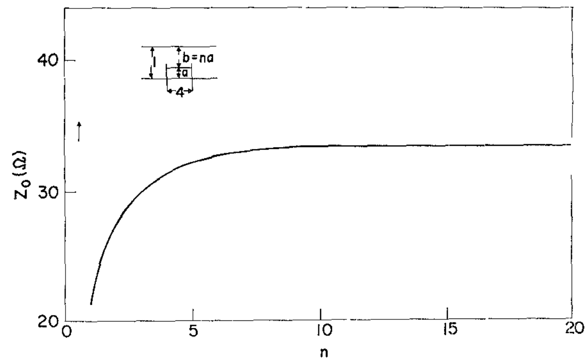

Fig. 5. Effect of upper ground plane separation.

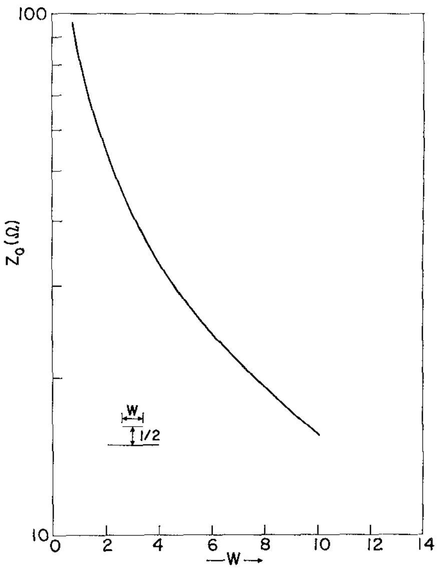

Fig. 6. Microstrip impedance as limit case of ground plane displacement.

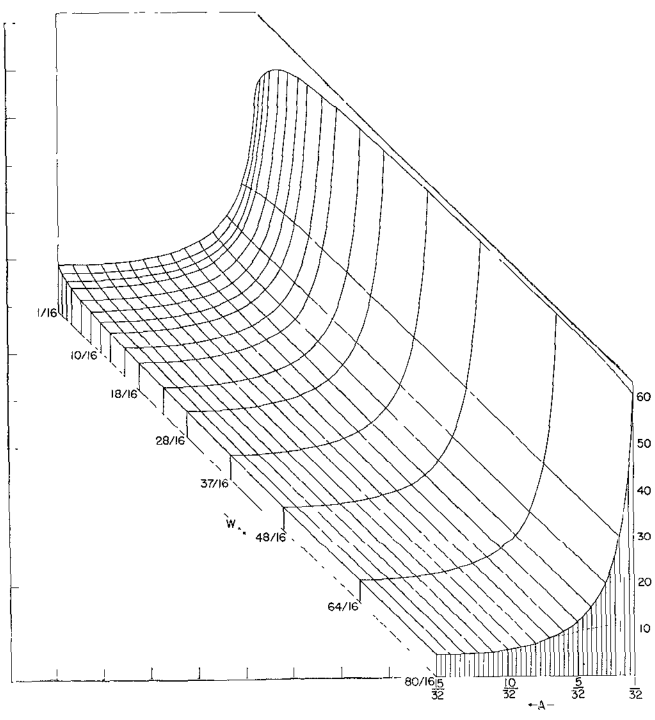

Fig. 7. Normalized loss $\bar{\alpha}=(2\eta d/R_S\sqrt{\epsilon R})\ \alpha_c$ for $T=0.032$ and $\epsilon_R=1$.

Conclusions

Since an upper and lower bound are calculated, the maximum possible error due to the approximations is always known. However, it was found that this error is overly pessimistic. Thus, the data given in Figs. 8 and 9 [4] indicates very good agreement with special cases solved by conformal mapping techniques. Further improvements can be obtained by changing the approximations for the lower bound. This has already been done for the upper bound. A solution based on the assumption that the surface charge is a constant was found to be too rough [6].

The method is very flexible. A single equation is used to solve a variety of problems, most of which have not been solved by transform methods. The inclusion of the moveable ground plane allows the evaluation of microstrip parameters in terms of the closed transmission line. Studies based on displacements, such as mechanical tolerances or losses, as indicated here, may be accomplished. It is, of course, true that the solutions are very complex. However, the higher order functions resulting from transformations have to be expanded in series form for purposes of evaluation anyway. Thus, as long as computer evaluation is used, little difference exists between the labor involved in the evaluation of the solutions and the greater utility of this type of approach which seems to justify the added complication.

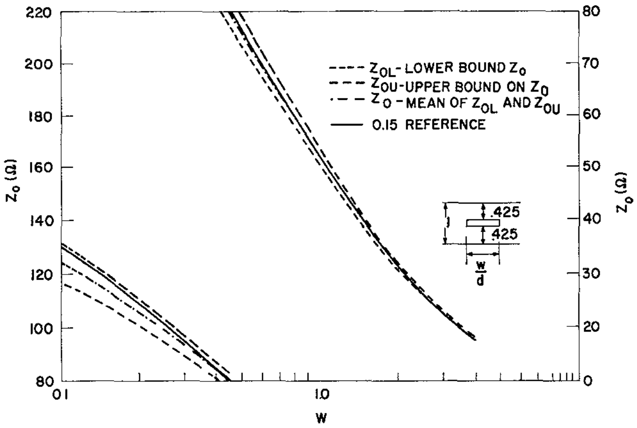

Fig. 8. Evaluation of the mathematical model by comparison to [4].

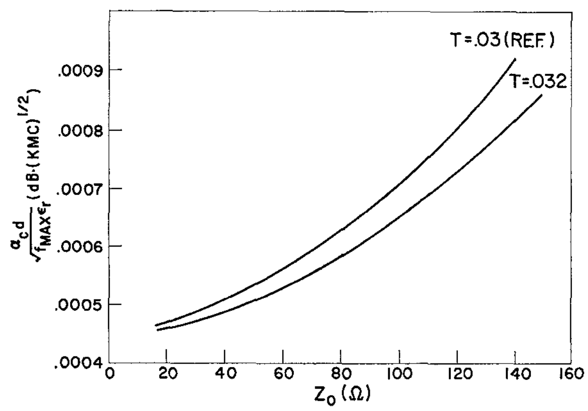

Fig. 9. Loss comparison with [4].

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